On a Formula of Reduction for Alternants of the Third Order.

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1. Denoting the simple alternant of the n^{th} order

$$\begin{vmatrix} a^p, & a^q, & \dots & a^t \\ b^p, & b^q, & \dots & b^t \\ \vdots & \vdots & & \vdots \\ l^p, & l^q, & \dots & l^t \end{vmatrix}$$

by $A(p, q, \dots t)$, the exponents, p, q, etc., being integers and all different, the quotient

$$\frac{A(p, q, \dots t)}{A(0, 1, \dots n-1)}$$

is a symmetric function of the n quantities $a, b, \dots t$, whose value was first expressed by Jacobi in the form of a determinant whose elements are complete symmetric functions of the n quantities $a, b, \dots l$. The denominator $A(0, 1, 2, \dots n-1)$ is the difference product

$$(b-a)(c-a)\dots(l-a).(c-b)\dots(l-b)\dots(k-l)$$

of the n quantities, or in Sylvester's notation $\zeta^{\frac{1}{2}}(a, b, c, \dots l)$; and if we denote the quotient by $\alpha(p, q, \dots t)$, Jacobi's result is expressed by the equation

$$(1) \qquad \alpha(p, q, \dots t) = \begin{vmatrix} H_p, & H_q, & \dots H_t \\ H_{p-1}, & H_{q-1}, & \dots H_{t-1} \\ \vdots & \vdots & \vdots \\ H_{p-n+1}, & H_{q-n+1}, & \dots H_{t-n+1} \end{vmatrix},$$

where H_p denotes the sum of all the powers and products of powers of the p^{th} degree, or what is the same thing, the sum of all the single symmetric functions Σa^p, $\Sigma a^p b$, etc., and it is to be understood that $H_0 = 1$ and $H_{-m} = 0$.*

* Mr. O. H. Mitchell has shown in this Journal that this result, obtained by Jacobi and others by an indirect process, is readily derived directly by elementary principles. *American Journal of Mathematics*, Vol. IV, p. 344 (December, 1881).

2. We need of course only consider alternants of which the lowest exponent is zero; thus, when the alternant is of the third order, we have

$$(2) \qquad \alpha(0, p, q) = \begin{vmatrix} 1, & H_p, & H_q \\ 0, & H_{p-1}, & H_{q-1} \\ 0, & H_{p-2}, & H_{q-2} \end{vmatrix} = H_{p-1}.H_{q-2} - H_{p-2}.H_{q-1};$$

but, even in this the simplest case, the expansion of the result in single symmetric functions is very laborious, the ordinary process producing, when p and q are moderately large, an enormous number of terms which cancel one another. The same is true to a great extent of the process given by Mr. Muir in his Treatise on Determinants, although this process shows that the first term of the result is $\Sigma a^{q-2}b^{p-1}$. The result is, however, readily obtained by means of the formula of reduction established below.

3. We have

$$A(0, p, q) = \begin{vmatrix} 1, & a^p, & a^q \\ 1, & b^p, & b^q \\ 1, & c^p, & c^q \end{vmatrix} = c^q(b^p - a^p) + a^q c^p - b^q(c^p - a^p) - a^q b^p.$$

Assuming $q > p$, by adding $0 = a^{q-p}b^p c^p - a^{q-p}b^p c^p$, this may be written in the form

$$\begin{aligned} A(0, p, q) &= c^q(b^p - a^p) - a^{q-p}c^p(b^p - a^p) - b^q(c^p - a^p) + a^{q-p}b^p(c^p - a^p) \\ &= c^p(b^p - a^p)(c^{q-p} - a^{q-p}) - b^p(c^p - a^p)(b^{q-p} - a^{q-p}); \end{aligned}$$

hence $\alpha(0, p, q) = \dfrac{A(0, p, q)}{(b-a)(c-a)(c-b)}$ is the quotient of

$$\begin{aligned} &c^p(b^{p-1} + b^{p-2}a + \ldots + a^{p-1})(c^{q-p-1} + c^{q-p-2}a + \ldots + a^{q-p-1}) \\ &\qquad - b^p(c^{p-1} + c^{p-2}a + \ldots + a^{p-1})(b^{q-p-1} + b^{q-p-2}a + \ldots + a^{q-p-1}) \end{aligned}$$

by $c - b$. Expanding the products, and grouping together similar positive and negative terms, we have, for the value of $(c-b)\,\alpha(0, p, q)$,

$$\begin{aligned} &b^{p-1}c^{p-1}(c^{q-p} - b^{q-p}) + ab^{p-2}c^{p-2}(c^{q-p+1} - b^{q-p+1}) + \ldots \\ &\qquad\qquad + a^{p-2}bc(c^{q-2} - b^{q-2}) + a^{p-1}(c^{q-1} - b^{q-1}) \\ &+ ab^{p-1}c^{p-1}(c^{q-p-1} - b^{q-p}) + a^2 b^{p-2}c^{p-2}(c^{q-p} - b^{q-p}) + \ldots \\ &\qquad\qquad + a^{p-1}bc(c^{q-3} - b^{q-3}) + a^p(c^{q-2} - b^{q-2}) \\ &\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ &+ a^{q-p-1}b^{p-1}c^{p-1}(c-b) + a^{q-p}b^{p-2}c^{p-2}(c^2 - b^2) + \ldots \\ &\qquad\qquad + a^{q-3}bc(c^{p-1} - b^{p-1}) + a^{q-2}(c^p - b^p), \end{aligned}$$

in which $p(q-p)$ binomials are written in $q-p$ rows and p columns. Replacing the binomials by their quotients by $c - b$, we have the expanded value of $\alpha(0, p, q)$.

Now, if we remove the first row and the last column of the rectangular array of binomials written above, we see that the remaining terms, when divided through by abc, constitute the value of $(c-b)\,\alpha(0,\ p-1,\ q-2)$. Hence
$\alpha(0,p,q)-abc.\alpha(0,p-1,q-2)=$
$$b^{p-1}c^{q-2}+b^pc^{q-3}+\ldots+b^{q-2}c^{p-1}+a(b^{p-2}c^{q-2}+b^{p-1}c^{q-3}+\ldots+b^{q-2}c^{p-2})+\ldots$$
$$+a^{p-1}(c^{q-2}+bc^{q-3}+\ldots+b^{q-2})+a^p(c^{q-3}+bc^{q-4}+\ldots+b^{q-3})+\ldots$$
$$+a^{q-2}(c^{p-1}+bc^{p-2}+\ldots+b^{p-1}).$$

It will be noticed that this is a symmetric function of a, b and c in which every product of the degree $q+p-3$ occurs once, except those in which there is an exponent greater than $q-2$. Denoting this function, which may be called a *curtailed* complete symmetric function, by $H_{q-2,\,p-1}$ (in which the sum of the suffixes indicates the degree), we have the formula of reduction

(3) $$\alpha(0,p,q)=H_{q-2,\,p-1}+abc.\alpha(0,p-1,q-2).$$

4. The formula may also be proved directly by division, as follows: we have $A(0,p,q)=c^q(b^p-a^p)+a^qc^p-$ (the result of interchanging b and c), and, in like manner, $abcA(0,p-1,q-2)=abc^{q-1}(b^{p-1}-a^{p-1})+a^{q-1}bc^p-$ (result of interchanging b and c); hence $A(0,\ p,\ q)-abcA(0,\ p-1,\ q-2)=c^q(b^p-a^p)-abc^{q-1}(b^{p-1}-a^{p-1})-a^{q-1}c^p(b-a)-$ (result of interchanging b and c).

If, therefore, we divide these terms by $(b-a)(c-a)$ and then subtract the result of interchanging b and c, we shall have the value of
$$(c-b)[\alpha(0,p,q)-abc.\alpha(0,p-1,q-2)].$$
Dividing the terms written above by $b-a$, we have
$$c^q(b^{p-1}+ab^{p-2}+\ldots+a^{p-1})-abc^{q-1}(b^{p-2}+ab^{p-3}+\ldots+a^{p-2})-a^{q-1}c^p$$
or $b^{p-1}c^{q-1}(c-a)+ab^{p-2}c^{q-1}(c-a)+\ldots$
$$+a^{p-2}bc^{q-1}(c-a)+a^{p-1}c^p(c^{q-p}-a^{q-p});$$
and, dividing this by $c-a$,
$$b^{p-1}c^{q-1}+ab^{p-2}c^{q-1}+\ldots+a^{p-2}bc^{q-1}+a^{p-1}c^p(c^{q-p-1}+ac^{q-p-2}+\ldots+a^{q-p-1}).$$
Finally, subtracting the result of interchanging b and c, we have
$$b^{p-1}c^{p-1}(c^{q-p}-b^{q-p})+ab^{p-2}c^{p-2}(c^{q-p+1}-b^{q-p+1})+\ldots+a^{p-2}bc(c^{q-2}-b^{q-2})$$
$$+a^{p-1}(c^{q-1}-b^{q-1})+a^p(c^{q-2}-b^{q-2})+\ldots+a^{q-2}(c^p-b^p)$$
as before, and dividing by $c-b$, the result is
$$\alpha(0,p,q)-abc.\alpha(0,p-1,q-2)=H_{q-2,\,p-1}.$$

5. When $p=1$, the formula reduces to

(4) $$\alpha(0,\ 1,\ q)=H_{q-2},$$

which results directly from Jacobi's theorem, equation (2); but, starting from this result, we may give an independent proof of the formula (3) as follows: If we write down all the terms in the complete symmetric function H_{r+s} for three quantities a, b and c in a triangular form a^{r+s}, b^{r+s} and c^{r+s} being the terms at the vertices, it is readily seen that the curtailed function $H_{r,s}$ is obtained by cutting off a small triangle of terms at each vertex, and that the terms in the first of these triangles are the same as those of the expression $a^{r+1}H_{s-1}$. Thus

(5) $$H_{r,s} = H_{r+s} - \Sigma a^{r+1}.H_{s-1},$$

in which the curtailed function is expressed in terms of complete symmetric functions. Putting $r = q - 2$ and $s = p - 1$,

$$H_{q-2,\,p-1} = H_{q+p-3} - \Sigma a^{q-1}.H_{p-2},$$

and, by equation (4),

$$H_{q-2,\,p-1} = \alpha(0,\ 1,\ q+p-1) - \Sigma a^{q-1}.\alpha(0,\ 1,\ p),$$

or $$\zeta^{\frac{1}{2}}(a,\ b,\ c).H_{q-2,\,p-1} = A(0,\ 1,\ q+p-1) - \Sigma a^{q-1}.A(0,\ 1,\ p)$$

$$= \begin{vmatrix} 1, & a, & a^{q+p-1} \\ 1, & b, & b^{q+p-1} \\ 1, & c, & c^{q+p-1} \end{vmatrix} - \begin{vmatrix} 1, & a, & a^p(a^{q-1}+b^{q-1}+c^{q-1}) \\ 1, & b, & b^p(a^{q-1}+b^{q-1}+c^{q-1}) \\ 1, & c, & c^p(a^{q-1}+b^{q-1}+c^{q-1}) \end{vmatrix} = - \begin{vmatrix} 1, & a, & a^p(b^{q-1}+c^{q-1}) \\ 1, & b, & b^p(c^{q-1}+a^{q-1}) \\ 1, & c, & c^p(a^{q-1}+b^{q-1}) \end{vmatrix}.$$

But, by the theorem in the preceding paper, this alternating function is equal to

$$-[1.b.a^{q-1}c^p + b^{q-1}c^p] = -[a^{q-1}bc^p + b^q c^p] = -A(q-1,\ 1,\ p) - A(0,\ q,\ p).$$

Hence $$\zeta^{\frac{1}{2}}(a,\ b,\ c).H_{q-2,\,p-1} = A(0,\ p,\ q) - A(1,\ p,\ q-1)$$
$$= A(0,\ p,\ q) - abcA(0,\ p-1,\ q-2),$$

or $$\alpha(0,\ p,\ q) = H_{q-2,\,p-1} + abc\alpha(0,\ p-1,\ q-2) \text{ as before.}$$

6. As an example of the use of the formula, let us find the value of $\alpha(0,\ 5,\ 7)$, that is the quotient of $A(0,\ 5,\ 7)$ by the difference product of the quantities a, b and c. We have

$$\alpha(0,\ 5,\ 7) = H_{5,\,4} + abc\alpha(0,\ 4,\ 5)$$
$$= H_{5,\,4} + abcH_{3,\,3},$$

since $\alpha(0,\ 3,\ 3)$ vanishes: or, writing out the values in single symmetric functions,

$$\alpha(0,\ 5,\ 7) = \Sigma a^5b^4 + \Sigma a^5b^3c + \Sigma a^5b^2c^2 + 2\Sigma a^4b^4c + 2\Sigma a^4b^3c^2 + 2\Sigma a^3b^3c^3.$$

Again, to find $\alpha(0,\ 3,\ 8)$, we have

$$\alpha(0,\ 3,\ 8) = H_{6,\,2} + abc\alpha(0,\ 2,\ 6)$$
$$= H_{6,\,2} + abcH_{4,\,1} + a^2b^2c^2\alpha(0,\ 1,\ 4)$$
$$= H_{6,\,2} + abcH_{4,\,1} + a^2b^2c^2H_2;$$

or, in single symmetric functions,

$$\alpha(0, 3, 8) = \Sigma a^6b^2 + \Sigma a^6bc + \Sigma a^5b^3 + \Sigma a^5b^2c + \Sigma a^4b^4 + \Sigma a^4b^3c + \Sigma a^4b^2c^2 + \Sigma a^3b^3c^2$$
$$+ abc[\Sigma a^4b + \Sigma a^3b^2 + \Sigma a^3bc + \Sigma a^2b^2c] + a^2b^2c^2[\Sigma a^2 + \Sigma ab]$$
$$= \Sigma a^6b^2 + \Sigma a^6bc + \Sigma a^5b^3 + 2\Sigma a^5b^2c + \Sigma a^4b^4 + 2\Sigma a^4b^3c + 3\Sigma a^4b^2c^2 + 3\Sigma a^3b^3c^2.$$

In the final expansion, it is to be noticed that, for a term in which the highest exponent (in this case 6) or the exponent zero occurs, the coefficient is unity; otherwise it is 2, provided the next higher or next lower exponent (5 or 1) occurs; but if not, it is 3, provided the next higher or next lower exponent (4 or 2) occurs, and so on. This is the general rule with, however, the restriction that no coefficient must exceed the number of H's in the development which is *the least of the numbers* p *and* $q - p$. The restriction takes effect whenever the last H has a suffix greater than 2, as in the first of the examples above.

7. In general

(6) $$\alpha(0, p, q) = H_{q-2,\, p-1} + abcH_{q-4,\, p-2} + a^2b^2c^2H_{q-6,\, p-3} + \text{etc.},$$

the series ending either with an H in which the two suffixes are equal, as in the first example above, or with one in which the second suffix is zero, that is, with a complete symmetric function, as in the second example.

The first of these cases corresponds to the theorem

$$\alpha(0, p+1, p+2) = H_{p,\, p},$$

a case of the more general theorem

(7) $$\alpha(0, p+1, p+2, \ldots p+n-1) = H_{p,\, p, \ldots}$$

where $H_{p,\, p, \ldots}$ denotes the symmetric function of n quantities in which every product of the degree $(n-1)p$ occurs once, except those in which there is an exponent greater than p. This theorem is readily derived from

(8) $$A(0, 1, 2, \ldots n-2, p+n-1) = \zeta^{\frac{1}{2}}(a, b, c, \ldots l).H_p$$

(which results directly from Jacobi's theorem) by substituting for the n quantities their reciprocals.

8. Equation (5), in which the curtailed symmetric function is expressed in terms of complete symmetric functions, holds for any number of quantities; hence by virtue of equation (8) the process of §5 is applicable to n quantities, the curtailed function being thus in general expressed as the sum of $n - 1$ co-factors of alternants. The result is, however, not generally available as a formula of reduction for an alternant: but, in the case of four quantities, if we put $r = s = p$ in equation (5), we have

$$H_{p,\, p} = H_{2p} - \Sigma a^{p+1}.\, H_{p-1};$$

whence $\zeta^{\frac{1}{2}}(a, b, c, d).H_{p,p} = A(0, 1, 2, 2p+3) - \Sigma a^{p+1}.A(0, 1, 2, p+2)$

$$= \begin{vmatrix} 1, & a, & a^2, & a^{2p+3} \\ 1, & b, & b^2, & b^{2p+3} \\ 1, & c, & c^2, & c^{2p+3} \\ 1, & d, & d^2, & d^{2p+3} \end{vmatrix} - \begin{vmatrix} 1, & a, & a^2, & a^{p+2}(a^{p+1}+b^{p+1}+c^{p+1}+d^{p+1}) \\ 1, & b, & b^2, & b^{p+2}(a^{p+1}+b^{p+1}+c^{p+1}+d^{p+1}) \\ 1, & c, & c^2, & c^{p+2}(a^{p+1}+b^{p+1}+c^{p+1}+d^{p+1}) \\ 1, & d, & d^2, & d^{p+2}(a^{p+1}+b^{p+1}+c^{p+1}+d^{p+1}) \end{vmatrix}$$

$$= -[1.b.c^2.d^{p+2}(a^{p+1}+b^{p+1}+c^{p+1})]$$

$$= -A(p+1, 1, 2, p+2) - A(0, p+2, 2, p+2) - A(0, 1, p+3, p+2).$$

Hence $\zeta^{\frac{1}{2}}(a, b, c, d)\,H_{p,p} = A(0, 1, p+2, p+3) - A(1, 2, p+1, p+2)$,
or

(9) $$\alpha(0, 1, p+2, p+3) = H_{p,p} + abcd.\alpha(0, 1, p, p+1),$$

a formula of reduction for an alternant of the form $A(0, 1, p+2, p+3)$. By repeated application of this formula, we have

(10) $$\alpha(0, 1, p+2, p+3) = H_{p,p} + abcdH_{p-2,p-2} + a^2b^2c^2d^2H_{p-4,p-4} + \text{etc.},$$

in which the last term is $(abcd)^{\frac{p}{2}}$ or $(abcd)^{\frac{p-1}{2}}\Sigma ab$, according as p is even or odd. For example,

$$\begin{vmatrix} 1, & a, & a^7, & a^8 \\ 1, & b, & b^7, & b^8 \\ 1, & c, & c^7, & c^8 \\ 1, & d, & d^7, & d^8 \end{vmatrix} \div \begin{vmatrix} 1, & a, & a^2, & a^3 \\ 1, & b, & b^2, & b^3 \\ 1, & c, & c^2, & c^3 \\ 1, & d, & d^2, & d^3 \end{vmatrix} = H_{5,5} + abcdH_{3,3} + a^2b^2c^2d^2H_{1,1}$$

$$= \Sigma a^5b^5 + \Sigma a^5b^4c + \Sigma a^5b^3c^2 + \Sigma a^5b^3cd + \Sigma a^5b^2c^2d + \Sigma a^4b^4c^2 + 2\Sigma a^4b^4cd$$
$$+ \Sigma a^4b^3c^3 + 2\Sigma a^4b^3c^2d + 2\Sigma a^4b^2c^2d^2 + 2\Sigma a^3b^3c^3d + 3\Sigma a^3b^3c^2d^2,$$

in which the coefficients follow the same rule as in §6, but no restriction need be observed with respect to the highest coefficient.